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A Study on Two Types of Fractional Integrals Involving Fractional Trigonometric Functions

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***Abstract*: In this paper, based on a new multiplication of fractional analytic functions, we study two types of fractional integrals involving fractional trigonometric functions. We can obtain the exact solutions of these two types of fractional integrals by using some techniques. Moreover, our results are generalizations of the results of ordinary calculus.**

***Keywords*: New multiplication, fractional analytic functions, fractional integrals, fractional trigonometric functions.**

I. INTRODUCTION

Fractional calculus is a branch of mathematical analysis, which studies several different possibilities of defining real order or complex order. In the second half of the 20th century, a large number of studies on fractional calculus were published in engineering literature. Fractional calculus is widely welcomed and concerned because of its applications in many fields such as mechanics, dynamics, control theory, physics, economics, viscoelasticity, electrical engineering, biology, and so on [1-11]

However, fractional calculus is different from ordinary calculus. The definition of fractional derivative is not unique. Common definitions include Riemann Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative and Jumarie's modification of R-L fractional derivative [12-16]. Because Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with classical calculus.

In this paper, based on a new multiplication of fractional analytic functions, we evaluate the following two types of fractional integrals involving fractional trigonometric functions:

$$\left({}_{0}I_{x}^{\alpha}\right)\left[r^{p}[(1-2rcos_{\alpha}(x^{\alpha})+r^{2})]^{\otimes_{\alpha}(-1)}\otimes_{\alpha}[cos_{\alpha}(px^{\alpha})-rcos_{\alpha}((p-1)x^{\alpha})]\right],$$

and

$$\left({}_{0}I_{x}^{\alpha}\right)\left[r^{p}[(1-2rcos_{\alpha}(x^{\alpha})+r^{2})]^{\otimes_{\alpha}(-1)}\otimes_{\alpha}[sin_{\alpha}(px^{\alpha})-rsin_{\alpha}((p-1)x^{\alpha})]\right],$$

where $0<\alpha\leq 1$,p is a positive integer, r is a real number, and $|r|<1$. The exact solutions of these two fractional integrals can be obtained by using some methods. In addition, our results are generalizations of the results in traditional calculus.

II. PRELIMINARIES

Firstly, we introduce the fractional calculus used in this paper.

Definition 2.1 ([17])**:** Let $0<\alpha\leq 1$, and x_0 be a real number. The Jumarie's modified Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_{x}^{\alpha}\right)[f(x)]=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt\ , \quad (1)$$

And the Jumarie type of Riemann-Liouville α-fractional integral is defined by

$$\left({}_{x_0}I_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}\frac{f(t)}{(x-t)^{1-\alpha}}dt\ , \tag{2}$$

where $\Gamma(\ \ )$ is the gamma function.

In the following, some properties of Jumarie type of R-L fractional derivative are introduced.

Proposition 2.2 ([18])**:** *If* α, β, x_0, c *are real numbers and* $\beta \geq \alpha > 0$, *then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{3}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[c] = 0. \tag{4}$$

Next, we introduce the definition of fractional analytic function.

Definition 2.3 ([19])**:** If x, x_0, and a_n are real numbers for all n, $x_0 \in (a, b)$, and $0 < \alpha \leq 1$. If the function $f_\alpha: [a, b] \to R$ can be expressed as an α-fractional power series, i.e., $f_\alpha(x^\alpha) = \sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. Furthermore, if $f_\alpha: [a, b] \to R$ is continuous on closed interval $[a, b]$ and it is α-fractional analytic at every point in open interval (a, b), then f_α is called an α-fractional analytic function on $[a, b]$.

Next, we introduce a new multiplication of fractional analytic functions.

Definition 2.4 ([20])**:** Let $0 < \alpha \leq 1$, and x_0 be a real number. If $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}, \tag{5}$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}\ . \tag{6}$$

Then we define

$$\begin{aligned} & f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}\otimes_\alpha \sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} \\ &= \sum_{n=0}^{\infty}\frac{1}{\Gamma(n\alpha+1)}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)(x-x_0)^{n\alpha}. \end{aligned} \tag{7}$$

Equivalently,

$$\begin{aligned} & f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty}\frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}\otimes_\alpha \sum_{n=0}^{\infty}\frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n} \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}. \end{aligned} \tag{8}$$

Definition 2.5 ([21])**:** If $0 < \alpha \leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} = \sum_{n=0}^{\infty}\frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}, \tag{9}$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} = \sum_{n=0}^{\infty}\frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}. \tag{10}$$

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The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha \circ g_\alpha)(x^\alpha) = f_\alpha\big(g_\alpha(x^\alpha)\big) = \sum_{n=0}^{\infty} \frac{a_n}{n!}\big(g_\alpha(x^\alpha)\big)^{\otimes_\alpha n}, \tag{11}$$

and

$$(g_\alpha \circ f_\alpha)(x^\alpha) = g_\alpha\big(f_\alpha(x^\alpha)\big) = \sum_{n=0}^{\infty} \frac{b_n}{n!}\big(f_\alpha(x^\alpha)\big)^{\otimes_\alpha n}. \tag{12}$$

Definition 2.6 ([22])**:** If $0 < \alpha \le 1$, and x is a real variable. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha n}. \tag{13}$$

And the α-fractional logarithmic function $Ln_\alpha(x^\alpha)$ is the inverse function of $E_\alpha(x^\alpha)$. On the other hand, the α-fractional cosine and sine function are defined as follows:

$$cos_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^k x^{2n\alpha}}{\Gamma(2n\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2n}, \tag{14}$$

and

$$sin_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (2n+1)}. \tag{15}$$

Definition 2.7 ([23])**:** Let $0 < \alpha \le 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ be two α-fractional analytic functions. Then $\big(f_\alpha(x^\alpha)\big)^{\otimes_\alpha n} = f_\alpha(x^\alpha)\otimes_\alpha \cdots \otimes_\alpha f_\alpha(x^\alpha)$ is called the nth power of $f_\alpha(x^\alpha)$. On the other hand, if $f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha) = 1$, then $g_\alpha(x^\alpha)$ is called the $\otimes_\alpha$ reciprocal of $f_\alpha(x^\alpha)$, and is denoted by $\big(f_\alpha(x^\alpha)\big)^{\otimes_\alpha (-1)}$.

Definition 2.8 ([24])**:** If $0 < \alpha \le 1$, and x is a real number. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha n}. \tag{16}$$

And the α-fractional logarithmic function $Ln_\alpha(x^\alpha)$ is the inverse function of $E_\alpha(x^\alpha)$. On the other hand, the α-fractional cosine and sine function are defined as follows:

$$cos_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n\alpha}}{\Gamma(2n\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2n}, \tag{17}$$

and

$$sin_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (2n+1)}. \tag{18}$$

Theorem 2.9 (fractional Euler's formula) ([25])**:** *If* $0 < \alpha \le 1$, *and* $i = \sqrt{-1}$, *then*

$$E_\alpha(ix^\alpha) = cos_\alpha(x^\alpha) + isin_\alpha(x^\alpha). \tag{19}$$

Theorem 2.10 (fractional DeMoivre's formula)([26])**:** *If* $0 < \alpha \le 1$, *and* p *is an integer, then*

$$[cos_\alpha(x^\alpha) + isin_\alpha(x^\alpha)]^{\otimes_\alpha p} = cos_\alpha(pAx^\alpha) + isin_\alpha(pAx^\alpha). \tag{20}$$

Notation 2.11: If the complex number $z = p + iq$, where p, q are real numbers, p the real part of z, is denoted by $\mathrm{Re}(z)$; q the imaginary part of z, is denoted by $\mathrm{Im}(z)$.

III. MAIN RESULTS

In this section, we will evaluate two types of fractional integrals involving fractional trigonometric functions. At first, a lemma is needed.

Lemma 3.1: *If* $0 < \alpha \le 1$, p *is a positive integer, and* r *is a real number, then*

$$\text{Re}\left\{[rE_\alpha(ix^\alpha)]^{\otimes_\alpha p}\otimes_\alpha [1-rE_\alpha(ix^\alpha)]^{\otimes_\alpha(-1)}\right\}$$
$$= r^p[(1-2r\cos_\alpha(x^\alpha)+r^2)]^{\otimes_\alpha(-1)}\otimes_\alpha [\cos_\alpha(px^\alpha)-r\cos_\alpha((p-1)x^\alpha)], \quad (21)$$
$$\text{Im}\left\{[rE_\alpha(ix^\alpha)]^{\otimes_\alpha p}\otimes_\alpha [1-rE_\alpha(ix^\alpha)]^{\otimes_\alpha(-1)}\right\}$$
$$= r^p[(1-2r\cos_\alpha(x^\alpha)+r^2)]^{\otimes_\alpha(-1)}\otimes_\alpha [\sin_\alpha(px^\alpha)-r\sin_\alpha((p-1)x^\alpha)]. \quad (22)$$

Proof Since $|r|<1$, it follows from fractional Euler's formula and fractional DeMoivre's formula that

$$[rE_\alpha(ix^\alpha)]^{\otimes_\alpha p}\otimes_\alpha [1-rE_\alpha(ix^\alpha)]^{\otimes_\alpha(-1)}$$
$$= r^pE_\alpha(ipx^\alpha)\otimes_\alpha \left[\left(1-r\cos_\alpha(x^\alpha)\right)-ir\sin_\alpha(x^\alpha)\right]^{\otimes_\alpha(-1)}$$
$$= r^p[\cos_\alpha(px^\alpha)+i\sin_\alpha(px^\alpha)]\otimes_\alpha [(1-2r\cos_\alpha(x^\alpha)+r^2)]^{\otimes_\alpha(-1)}\otimes_\alpha \left[\left(1-r\cos_\alpha(x^\alpha)\right)+ir\sin_\alpha(x^\alpha)\right]. \quad (23)$$

Thus,

$$\text{Re}\left\{[rE_\alpha(ix^\alpha)]^{\otimes_\alpha p}\otimes_\alpha [1-rE_\alpha(ix^\alpha)]^{\otimes_\alpha(-1)}\right\}$$
$$= r^p[(1-2r\cos_\alpha(x^\alpha)+r^2)]^{\otimes_\alpha(-1)}\otimes_\alpha \left[\cos_\alpha(px^\alpha)\otimes_\alpha\left(1-r\cos_\alpha(x^\alpha)\right)-\sin_\alpha(px^\alpha)\otimes_\alpha r\sin_\alpha(x^\alpha)\right]$$
$$= r^p[(1-2r\cos_\alpha(x^\alpha)+r^2)]^{\otimes_\alpha(-1)}\otimes_\alpha [\cos_\alpha(px^\alpha)-r\cos_\alpha((p-1)x^\alpha)].$$

Similarly,

$$\text{Im}\left\{[rE_\alpha(ix^\alpha)]^{\otimes_\alpha p}\otimes_\alpha [1-rE_\alpha(ix^\alpha)]^{\otimes_\alpha(-1)}\right\}$$
$$= r^p[(1-2r\cos_\alpha(x^\alpha)+r^2)]^{\otimes_\alpha(-1)}\otimes_\alpha \left[\cos_\alpha(px^\alpha)\otimes_\alpha r\sin_\alpha(x^\alpha)+\sin_\alpha(px^\alpha)\otimes_\alpha\left(1-r\cos_\alpha(x^\alpha)\right)\right]$$
$$= r^p[(1-2r\cos_\alpha(x^\alpha)+r^2)]^{\otimes_\alpha(-1)}\otimes_\alpha [\sin_\alpha(px^\alpha)-r\sin_\alpha((p-1)x^\alpha)].$$ q.e.d.

Theorem 3.2: *If* $0<\alpha\le 1$, p *is a positive integer,* r *is a real number, and* $|r|<1$, *then the* α*-fractional integrals*

$$\left({}_0I_x^\alpha\right)\left[r^p[(1-2r\cos_\alpha(x^\alpha)+r^2)]^{\otimes_\alpha(-1)}\otimes_\alpha [\cos_\alpha(px^\alpha)-r\cos_\alpha((p-1)x^\alpha)]\right]$$
$$= \sum_{n=0}^{\infty} r^{n+p}\frac{1}{n+p}\sin_\alpha((n+p)x^\alpha),$$

(24)

and

$$\left({}_0I_x^\alpha\right)\left[r^p[(1-2r\cos_\alpha(x^\alpha)+r^2)]^{\otimes_\alpha(-1)}\otimes_\alpha [\sin_\alpha(px^\alpha)-r\sin_\alpha((p-1)x^\alpha)]\right]$$
$$= -\sum_{n=0}^{\infty} r^{n+p}\frac{1}{n+p}\cos_\alpha((n+p)x^\alpha).$$

(25)

Proof Since $|r|<1$, it follows from fractional DeMoivre's formula that

$$[rE_\alpha(ix^\alpha)]^{\otimes_\alpha p}\otimes_\alpha [1-rE_\alpha(ix^\alpha)]^{\otimes_\alpha(-1)}$$
$$= r^pE_\alpha(ipx^\alpha)\otimes_\alpha \sum_{n=0}^{\infty}[rE_\alpha(ix^\alpha)]^{\otimes_\alpha n}$$
$$= r^pE_\alpha(ipx^\alpha)\otimes_\alpha \sum_{n=0}^{\infty} r^nE_\alpha(inx^\alpha)$$
$$= \sum_{n=0}^{\infty} r^{n+p}E_\alpha(i(n+p)x^\alpha). \quad (26)$$

Therefore, by Lemma 3.1

$$\left({}_0I_x^{\alpha}\right)\left[r^p[(1-2r\cos_\alpha(x^\alpha)+r^2)]^{\otimes_\alpha(-1)}\otimes_\alpha[\cos_\alpha(px^\alpha)-r\cos_\alpha((p-1)x^\alpha)]\right]$$

$$=\left({}_0I_x^{\alpha}\right)\left[\operatorname{Re}\left\{[rE_\alpha(ix^\alpha)]^{\otimes_\alpha p}\otimes_\alpha[1-rE_\alpha(ix^\alpha)]^{\otimes_\alpha(-1)}\right\}\right]$$

$$=\left({}_0I_x^{\alpha}\right)\left[\operatorname{Re}\{\textstyle\sum_{n=0}^{\infty}r^{n+p}E_\alpha(i(n+p)x^\alpha)\}\right]$$

$$=\left({}_0I_x^{\alpha}\right)\left[\textstyle\sum_{n=0}^{\infty}r^{n+p}\cos_\alpha((n+p)x^\alpha)\right]$$

$$=\textstyle\sum_{n=0}^{\infty}r^{n+p}\left({}_0I_x^{\alpha}\right)[\cos_\alpha((n+p)x^\alpha)]$$

$$=\textstyle\sum_{n=0}^{\infty}r^{n+p}\frac{1}{n+p}\sin_\alpha((n+p)x^\alpha)\,.$$

And

$$\left({}_0I_x^{\alpha}\right)\left[r^p[(1-2r\cos_\alpha(x^\alpha)+r^2)]^{\otimes_\alpha(-1)}\otimes_\alpha[\sin_\alpha(px^\alpha)-r\sin_\alpha((p-1)x^\alpha)]\right]$$

$$=\left({}_0I_x^{\alpha}\right)\left[\operatorname{Im}\left\{[rE_\alpha(ix^\alpha)]^{\otimes_\alpha p}\otimes_\alpha[1-rE_\alpha(ix^\alpha)]^{\otimes_\alpha(-1)}\right\}\right]$$

$$=\left({}_0I_x^{\alpha}\right)\left[\operatorname{Im}\{\textstyle\sum_{n=0}^{\infty}r^{n+p}E_\alpha(i(n+p)x^\alpha)\}\right]$$

$$=\left({}_0I_x^{\alpha}\right)\left[\textstyle\sum_{n=0}^{\infty}r^{n+p}\sin_\alpha((n+p)x^\alpha)\right]$$

$$=\textstyle\sum_{n=0}^{\infty}r^{n+p}\left({}_0I_x^{\alpha}\right)[\sin_\alpha((n+p)x^\alpha)]$$

$$=-\textstyle\sum_{n=0}^{\infty}r^{n+p}\frac{1}{n+p}\cos_\alpha((n+p)x^\alpha)\,.$$

q.e.d.

IV. CONCLUSION

In this paper, based on a new multiplication of fractional analytic functions, two fractional integrals involving fractional trigonometric functions are studied. Using some methods, we can find the exact solutions of these two fractional integrals. Moreover, our results are generalizations of classical calculus results. In the future, we will continue to use the new multiplication of fractional analytic functions to solve the problems in fractional differential equations and applied mathematics.

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